Note first that $\mathcal{F}$ lies in $C(K)$: it is clear from the definition of $\mathcal{F}$ that every element of $\mathcal{F}$ is continuous. Note as well that $\mathcal{F}$ is contained in $\prod_{x\in K}\overline{D}(0, r(x))$, where $\overline{D}(0, r(x))$ is the closed disk in the complex plane with center 0 and radius $r(x)$. Let $\mathcal{F}_1$ denote $\mathcal{F}$ with the topology of uniform convergence, and let $\mathcal{F}_2$ denote $\mathcal{F}$ with the topology it inherits as a subspace of $\prod_{x\in K}\overline{D}(0, r(x))$. We will show that $\mathcal{F}_2$ is a closed subspace of $\prod_{x\in K}\overline{D}(0, r(x))$ and that the identity map from $\mathcal{F}_2$ to $\mathcal{F}_1$ is continuous. It will then follow from the Tychonoff theorem that $\mathcal{F}_2$ (and hence $\mathcal{F}_1$) is compact.

The fact that $\mathcal{F}_2$ is a closed subset of $\prod_{x\in K}\overline{D}(0, r(x))$ is clear: if $< f_\alpha >_{\alpha\in A}$ is a net in $\mathcal{F}_2$ converging to f in $\prod_{x\in K}\overline{D}(0, r(x))$ then $f_\alpha(x) \to f(x)$ for each x in K, so f belongs to $\mathcal{F}_2$.

The proof that the identity mapping from $\mathcal{F}_2$ to $\mathcal{F}_1$ is continuous is proved as in the traditional proof of the Ascoli-Arzelà Theorem: Suppose that $\epsilon > 0$. The fact that K is compact shows that there exist finitely many points $x_1, \ldots x_n$ of K such that $K = \bigcup_{j=1}^{n} \omega(x_j, \epsilon)$. Now if $< f_\alpha >_{\alpha\in A}$ is a net converging to f in $\mathcal{F}_2$ then there exists α_0 in A such that $|f_\alpha(x_j) - f(x_j)| < \epsilon$ for $j = 1, \ldots, n$ whenever $\alpha \geq \alpha_0$. Now for any such α and any x in K we may choose j in $\{1, \ldots, n\}$ such that x belongs to $\omega(x_j, \epsilon)$. It follows that

$$|f_\alpha(x) - f(x)| \leq |f_\alpha(x) - f_\alpha(x_j)| + |f_\alpha(x_j) - f(x_j)| + |f(x_j) - f(x)| < 3\epsilon.$$

We conclude that the supremum of $|f_\alpha - f|$ is no larger than 3ϵ whenever $\alpha \geq \alpha_0$. ∎

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Klee's Trigonometry Problem

PL. Kannappan

1. INTRODUCTION. V. L. Klee posed the following problem in this MONTHLY [**6**].

Problem. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y) \tag{1}$$

for x and y in $\mathbb{R}$, and that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that f and g satisfy

$$g(x + y) = g(x)g(y) - f(x)f(y) \tag{2}$$

and

$$f(x \pm y) = f(x)g(y) \pm g(x)f(y) \tag{3}$$

for all real x and y.

A solution by T. S. Chihara appeared in this MONTHLY [**3**], but it unfortunately had a gap. We first determine the general solution of (1) wihout any additional conditions and obtain (2) and (3) in the process. We then give a simple and direct solution to Klee's problem with the added conditions.

Remark. Compare (1)–(3) with the familiar trigonometric formulas

$$\cos(x \mp y) = \cos x \cos y \pm \sin x \sin y,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

2. SOLUTION OF (1) ON GROUPS. Let $f, g : G \to \mathbb{C}$ (where G is an Abelian, two divisible group, and $\mathbb{C}$ denotes the field of complex numbers) satisfy (1) for x and y in G. (A group G is *two divisible* if each x in G can be expressed in the form $x = y + y = 2y$ for some (unique) y in G. Thus $x/2$ is meaningful for x in G.) We determine the general solution of (1) (see [**1**], [**2**], [**4**], [**5**], [**7**], [**9**], [**10**], and [**11**]).

First we consider constant solutions of (1). If $g \equiv c$, a constant, and $f \not\equiv 0$, then (1) gives that $f \equiv d$, a constant for which $d^2 + c^2 = c$, but this won't satisfy (2). If $f \equiv d$, a constant, then (1) becomes

$$g(x - y) = g(x)g(y) + d^2 = g(y - x).$$

In particular, g is even (i.e., $g(-x) = g(x)$), so $g(x - y) = g(x + y)$ (replace y by $-y$ in the foregoing identity and appeal to the evenness of g). Then with $x = (u + v)/2$ and $y = (u - v)/2$ for any u and v in G, we have $g(v) = g(u)$, that is, g is a constant (here the two divisibility of G is used). Identity (2) won't hold for such solutions unless $d = 0 = c$.

We henceforth consider only the nonconstant (nontrivial) solutions f and g of (1). Interchange of x and y in (1) yields

$$g(x - y) = g(y - x),$$

whence g is even.

Replace y by $-y$ in (1) to get

$$g(x + y) = g(x)g(y) + f(x)f(-y) \tag{4}$$

for x and y in G. We show that f is odd (i.e., $f(-x) = -f(x)$). Change x to $-x$ in (1) to obtain

$$g(x + y) = g(x)g(y) + f(-x)f(y) \tag{5}$$

for x and y in G. Thus

$$f(x)f(-y) = f(-x)f(y).$$

Since $f \not\equiv 0$,

$$f(x) = kf(-x) = k^2 f(x)$$

for all x in G, where k is a constant such that $k^2 = 1$.

If $k = 1$, then f is even and (1) and (4) imply that $g(x - y) = g(x + y)$ and then g is constant (here again the two divisibility of G is used), contrary to assumption.

Hence $k = -1$, f is odd, and (4) becomes

$$g(x+y) = g(x)g(y) - f(x)f(y),$$

which is (2), and g satisfies the *cosine equation* (6) (add (1) and (2)):

$$g(x+y) + g(x-y) = 2g(x)g(y). \tag{6}$$

Further, applying associativity and (2), we get

$$g(x+y+z) = \left[g(x)g(y) - f(x)f(y)\right]g(z) - f(x+y)f(z)$$

and

$$g(x+y+z) = g(x)\left[g(y)g(z) - f(y)f(z)\right] - f(x)f(y+z),$$

that is,

$$\left[f(x+y) - f(y)g(x)\right]f(z) = \left[f(y+z) - f(y)g(z)\right]f(x)$$

or

$$f(x+y) - f(y)g(x) = h(y)f(x), \tag{7}$$

where

$$h(y) = \frac{1}{f(z_0)}\left[f(y+z_0) - f(y)g(z_0)\right]$$

with $f(z_0) \neq 0$.

Change x to $-x$ in (7) and use the fact that f is odd and g even to conclude that

$$f(x-y) = -f(y)g(x) + f(x)h(y), \tag{8}$$

that is (add (7) and (8)),

$$f(x+y) + f(x-y) = 2f(x)h(y) \tag{9}$$

for x and y in G. Interchange x and y in (9) to see that

$$f(x+y) - f(x-y) = 2f(y)h(x) \tag{10}$$

for x and y in G. Addition of (9) and (10) yields

$$f(x+y) = f(x)h(y) + f(y)h(x). \tag{11}$$

Since $f \not\equiv 0$, (7) and (11) imply that $h(x) = g(x)$ for all x in G. Hence (7) becomes

$$f(x+y) = f(x)g(y) + f(y)g(x) \tag{12}$$

for x and y in G, which is one part of (3). Replace y by $-y$ in (12) to obtain

$$f(x-y) = f(x)g(y) - f(y)g(x), \tag{13}$$

which is the other part of (3).

Now we are in a position to determine f and g. Since g satisfies the cosine equation (6), we must have

$$g(x) = \frac{E(x) + E^*(x)}{2} \tag{14}$$

for x in G, where $E : G \to \mathbb{C}^*$ (the nonzero complex numbers) is a homomorphism satisfying the exponential equation $E(x + y) = E(x)E(y)$ and $E^* = 1/E$ [**4**].

Inserting the representation (14) for g into (1) results, after a straightforward computation, in

$$f(x)f(y) = \frac{E(x) - E^*(x)}{2} \cdot \frac{E(y) - E^*(y)}{2}$$

for x and y in G. Since $f \not\equiv 0$,

$$f(x) = b\big(E(x) - E^*(x)\big) \tag{15}$$

for x in G, with $b^2 = 1/4$. Thus we have proved the following theorem:

Theorem 1. *Suppose that $f, g : G \to \mathbb{C}$ are nonconstant solutions of* (1), *where G is a two divisible Abelian group. Then f and g satisfy* (2) *and* (3). *Moreover, these functions are given by* (14) *and* (15) *for some homomorphism $E : G \to \mathbb{C}^*$.*

Theorem 1 has the following corollary (see [**4**], [**1**], and [**7**]):

Corollary 1. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be nonconstant solutions of* (1) *with g continuous. Then f is also continuous, and $g(x) = \cos(bx)$ and $f(x) = k\sin(bx)$, where $k^2 = 1/4$ and b is a complex constant.*

Remark. As a special case, Theorem 1 provides a solution to Klee's problem and shows that the additional conditions $f(t) = 1$ and $g(t) = 0$ are superfluous.

3. SELF-CONTAINED SOLUTION TO KLEE'S PROBLEM. We return briefly to the original problem posed by Klee and provide a solution that makes no appeal to [**4**].

Let $f, g : \mathbb{R} \to \mathbb{R}$ be solutions of (1). The assumption that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$ forces f and g to be nonconstant. As earlier, (1) implies that g is even.

Substituting $y = t$ in (1) gives

$$g(x - t) = f(x), \;\; f(x + t) = g(x), \tag{16}$$

so $f(-x) = g(x + t)$ for all real x. Now by (16) and (1),

$$\begin{aligned} f(x - y) &= g(x - y - t) \\ &= g(x)g(y + t) + f(x)f(y + t) \\ &= g(x)f(-y) + f(x)g(y) \end{aligned} \tag{17}$$

for x and y in $\mathbb{R}$. Change y to $-y$ in (17) to obtain the "+" half of (3). To verify both the other half of (3) and (2) it is enough to show that f is odd.

Arguing as before, we find that (4) holds, and from it we again infer that

$$f(x)f(-y) = f(y)f(-x)$$

for all real x and y.

Now setting $y = t$ and $x = -t$ yields $f(-t)^2 = 1$, so $f(-t) = \pm 1$. The choice $f(-t) = 1$ leads to the conclusion that f is even and g is constant, which is not the case. Thus f is odd and (17) and (4) become the "$-$" half of (3) and (2), respectively. This furnishes a solution to Klee's problem. Note that we have used the conditions $f(t) = 1$, $g(t) = 0$ a couple of times.

As remarked at the end of the solution of E1079, the usual formula for $\cos(x \pm y)$ and $\sin(x \pm y)$ follow purely algebraically from the formula for $\cos(x - y)$.

ACKNOWLEDGMENT This paper is dedicated to Professor Z. Moszner on the occasion of his 73rd birthday.

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Green's Theorem and the Fundamental Theorem of Algebra

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One proof of the fundamental theorem of algebra uses Liouville's theorem, which follows from Cauchy's theorem, which in turn can be derived from Green's theorem; see, for instance, the beautiful book [**1**]. The purpose of this note is to show that Green's theorem is sufficient. The proof does not use any topology or analytic function theory.